Fractions, II

September 5, 2013

Wichita, Kansas

H. Wu

Part I. Review

Part II. Applications of multiplication

Part III. Division

Part IV. Complex fractions

Part V. Percent

Part VI. Ratio

Part VII. Constant rate

PART I. REVIEW

1. Everything we do here is consistent with the CCSSM.

H. Wu, (2011). *Understanding Numbers in Elementary School Mathematics.* Providence, RI. American Mathematical Society.

H. Wu, (2011). Teaching Fractions According to the Common Core Standards.
http://math.berkeley.edu/~wu/CCSS-Fractions.pdf.

2. We give precise definitions for all concepts.

3. We give reason for every assertion.

4. The arithmetic operations of fractions are very similar to those of whole numbers.

5. Content coverage in the afternoon session:

Applications of multiplication, division, complex fractions, percent, ratio, constant rate.

This presentation directly addresses the teaching of fractions in the 5th and 6th grades, but the basic idea of how to approach fractions is of course applicable to all of K-12.

6. What are we really fighting against?

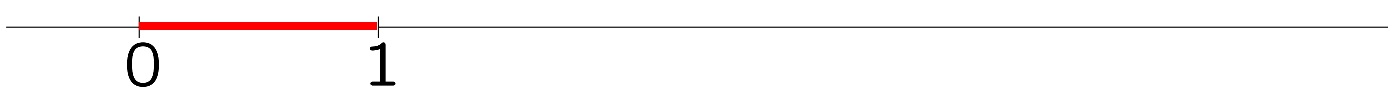
The mathematics encoded in existing textbooks, e.g., absence of definition of a fraction, how equivalent fractions is taught, etc.

We call this body of flawed knowledge,

TSM (Textbook School Mathematics).

The **number line**.

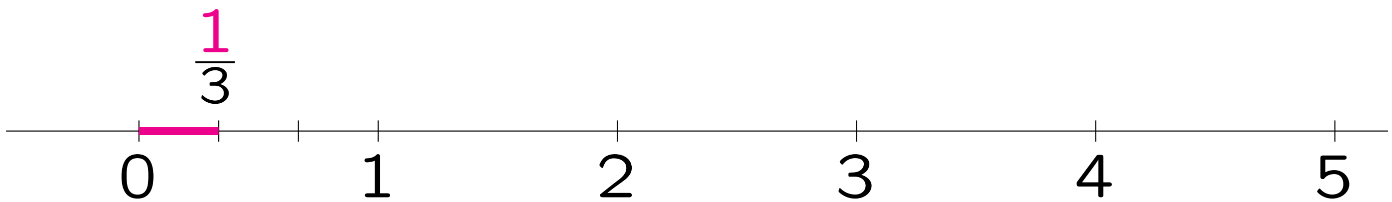
On a horizontal line, let two points be singled out. Identify the point to the left with 0 and the one to the right with 1. This segment, denoted by **[0,1]** is called the **unit segment** and 1 is called the **unit.**



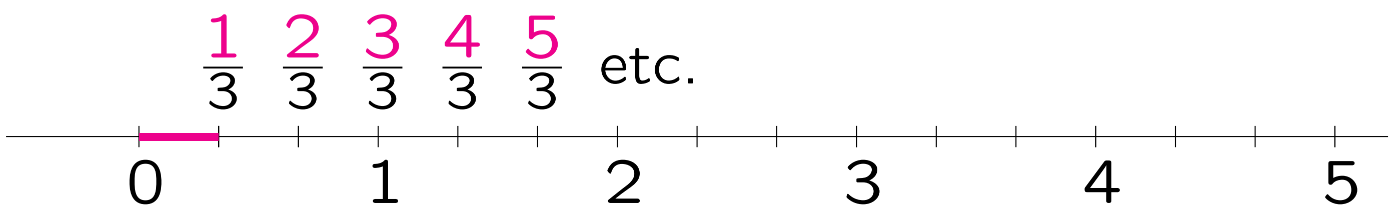
Now mark off equidistant points to the right of 1 as in a ruler, as shown, and identify the successive points with 2, 3, 4, ...

The line, with a sequence of equidistant points on the right identified with the whole numbers, is called the **number line**.

Divide [0,1] into three segments of *equal length*. The part adjoining 0 is a third. Denote its right endpoint by $\frac{1}{3}$.

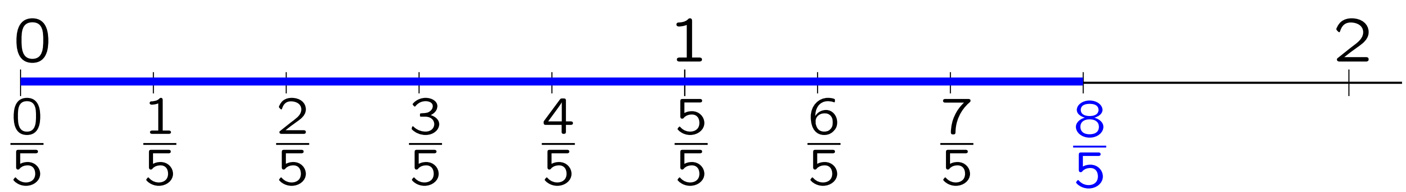


Fix the distance between 0 and $\frac{1}{3}$. Marking off *equidistant* points to the right of $\frac{1}{3}$ as we would with whole numbers, we obtain a sequence of points, denoted by $\frac{2}{3}$, $\frac{3}{3}$, $\frac{4}{3}$, etc.



This gives all the fractions with denominator equal to 3.

Fractions with denominator equal to 5 are similarly placed on the number line: $\frac{8}{5}$ is the 8th point to the right of 0 in the sequence of *fifths*. And so on.



We also agree to identify $\frac{0}{n}$ *with* 0 *for any nonzero whole number* n*.* In this way, all fractions are unambiguously placed on the number line.

Intuitively, we have identified *parts-of-a-whole* with points on the number line.

(Finite) decimals

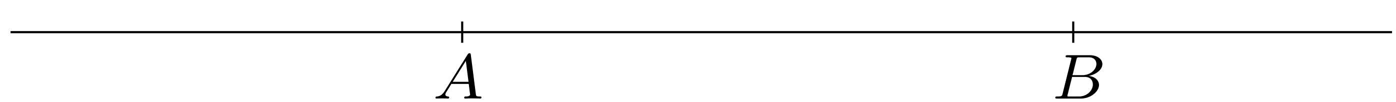
They are fractions whose denominators are 1, 10, 100, 1000, ... Back in 1593, the German Jesuit astronomer C. Clavius (1538-1612) introduced the special notation below:

$$\frac{235}{100}\ \left(=\frac{235}{10^2}\right)\ \text{ is simplified to }\ 2.35;$$

$$\frac{57}{10000}\ \left(=\frac{57}{10^4}\right)\ \text{ is simplified to }\ 0.0057$$

The first major benefit of having a precise definition of a fraction: we can now define precisely what it means for two fractions A and B to be **equal**, or for A to be **smaller than** B:

By definition, $\boldsymbol{A = B}$ if A and B are the same point on the number line, and $\boldsymbol{A < B}$ if the point A on the number line is to the left of B:



The pivotal theorem.

Theorem on equivalent fractions. *Given any fractions $\frac{k}{\ell}$ and a nonzero whole number c, then:*

$$\frac{k}{\ell} = \frac{c\,k}{c\,\ell}$$

This says that the two fractions $\frac{k}{\ell}$ and $\frac{c\,k}{c\,\ell}$ are the same point on the number line.

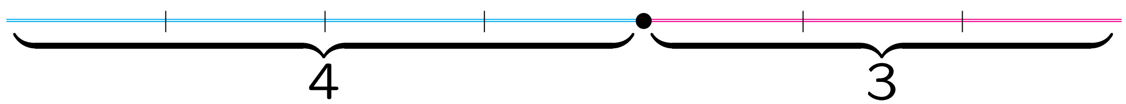
Fundamental Fact of Fraction Pairs (FFFP)

Any two fractions may be regarded as two fractions with the same denominator.

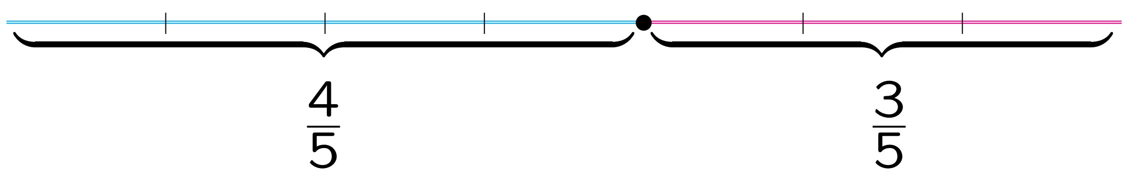
Given $\frac{m}{n}$ and $\frac{k}{\ell}$, they can be written as, respectively,

$$\frac{\ell m}{\ell n} \quad \text{and} \quad \frac{kn}{\ell n}$$

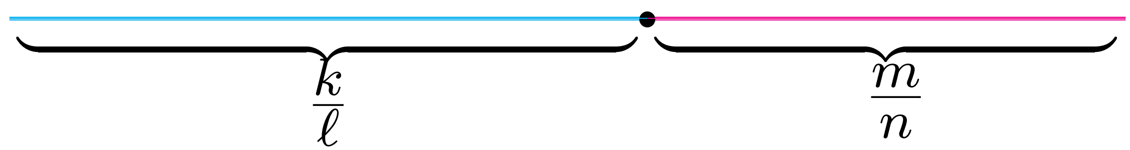
Addition: First, how do we add whole numbers, $3+4$?



How do we add $\frac{4}{5} + \frac{3}{5}$?



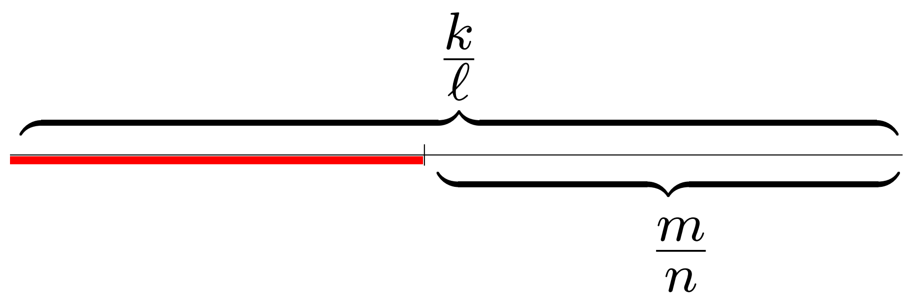
In general, we define the addition of $\frac{k}{\ell}$ and $\frac{m}{n}$ in exactly the same way: $\boldsymbol{\frac{k}{\ell}} + \boldsymbol{\frac{m}{n}}$ is the length of the concatenation of one segment of length $\frac{k}{\ell}$ and another of length $\frac{m}{n}$:



By FFFP,

$$\frac{k}{\ell} + \frac{m}{n} = \frac{kn}{\ell n} + \frac{\ell m}{\ell n} = \frac{kn + \ell m}{\ell n}$$

Subtraction: If $\frac{m}{n} < \frac{k}{\ell}$, then we define $\boldsymbol{\frac{k}{\ell} - \frac{m}{n}}$ to be the length of the remaining segment when a segment of length $\frac{m}{n}$ is removed from one end of a (longer) segment of length $\frac{k}{\ell}$.



$$\frac{k}{\ell} - \frac{m}{n} = \frac{kn}{\color{blue}{\ell n}} - \frac{\ell m}{\color{blue}{\ell n}} = \frac{kn - \ell m}{\ell n}$$

Multiplication: $\boldsymbol{\frac{k}{\ell} \times \frac{m}{n}}$ is the total length of k parts when $[0, \frac{m}{n}]$ is divided into ℓ equal parts.

The "total length of k parts when $[0, \frac{m}{n}]$ is divided into ℓ equal parts" is what we call "$\frac{k}{\ell}$ of $\frac{m}{n}$" in everyday language. Therefore,

$$\frac{k}{\ell} \text{ of } \frac{m}{n} \quad = \quad \frac{k}{\ell} \times \frac{m}{n}.$$

Because $\frac{1}{\ell} \times \frac{m}{n} = \frac{1}{\ell} \times \frac{\ell m}{\ell n} = \frac{m}{\ell n}$, we get

$$\frac{k}{\ell} \times \frac{m}{n} = \underbrace{\frac{m}{\ell n} + \cdots + \frac{m}{\ell n}}_{k \text{ times}}$$

Therefore,

$$\frac{k}{\ell} \times \frac{m}{n} = \frac{km}{\ell n}$$

This is the all important **Product Formula**.

PART II. APPLICATIONS OF MULTIPLICATION

The Product Formula shows that fraction multiplication is commutative, i.e.,

$$\frac{k}{\ell} \times \frac{m}{n} \;=\; \frac{m}{n} \times \frac{k}{\ell},$$

because

$$\frac{km}{\ell n} \;=\; \frac{mk}{n\ell}.$$

You think this formula is boring,? Try this:

Which is heavier:

$\frac{7}{9}$ of $\frac{11}{4}$ kg of sand, or $\frac{11}{4}$ of $\frac{7}{9}$ kg of sand?

(By definition, the first is the totality of 7 parts when $\frac{11}{4}$ is divided into 9 equal parts, while the latter is 11 parts when $\frac{7}{9}$ is divided into 4 equal parts.)

Fraction Multiplication is also associative and distributive in general:

$$\left(\frac{k}{\ell} \times \frac{m}{n}\right) \times \frac{a}{b} \;=\; \frac{k}{\ell} \times \left(\frac{m}{n} \times \frac{a}{b}\right)$$

and

$$\frac{k}{\ell} \times \left(\frac{m}{n} + \frac{a}{b}\right) \;=\; \left(\frac{k}{\ell} \times \frac{m}{n}\right) \;+\; \left(\frac{k}{\ell} \times \frac{a}{b}\right)$$

The verification using the Product Formula is routine (and somewhat tedious).

There are many more consequences of the Product Formula.

The **cancellation phenomenon**, e.g.,

$$\frac{\not{8} \times 5}{\not{9} \times 13} \times \frac{7 \times \not{9}}{\not{8} \times 11} = \frac{5}{13} \times \frac{7}{11}$$

i.e., *we cancelled the 8 in top and bottom, and cancelled the 9 in top and bottom.*

We could do that because, by the Product Formula and the theorem on equivalent fractions, we have:

$$\frac{8 \times 5}{9 \times 13} \times \frac{7 \times 9}{8 \times 11} = \frac{(8 \times 9) \times (5 \times 7)}{(8 \times 9) \times (13 \times 11)} = \frac{5 \times 7}{13 \times 11}$$

Obviously $\frac{5}{13} \times \frac{7}{11}$ is also equal to $\frac{5 \times 7}{13 \times 11}$.

By cancellation, any nonzero fraction $\frac{m}{n}$ satisfies $\frac{n}{m} \times \frac{m}{n} = 1$. Let $\frac{k}{\ell}$ be a fraction. Multiply both sides of the equality by $\frac{k}{\ell}$ to get: $\frac{k}{\ell} \times \left(\frac{n}{m} \times \frac{m}{n}\right) = \frac{k}{\ell}$.

By the associative law, we have

$$\left(\frac{k}{\ell} \times \frac{n}{m}\right) \times \frac{m}{n} = \frac{k}{\ell}$$

Denoting the fraction $\frac{k}{\ell} \times \frac{n}{m}$ by Q, this means:

Given any nonzero $\frac{m}{n}$ and any $\frac{k}{\ell}$, there is always a fraction Q so that $\frac{k}{\ell} = Q \times \frac{m}{n}$.

We saw that if $Q = \frac{k}{\ell} \times \frac{n}{m}$, then $\frac{k}{\ell} = Q \times \frac{m}{n}$. But could there be another fraction Q' so that we also have

$$\frac{k}{\ell} = Q' \times \frac{m}{n}\,?$$

The answer is no, because if we multiply both sides by $\frac{n}{m}$, we would get

$$\frac{k}{\ell} \times \frac{n}{m} = Q' \times \frac{m}{n} \times \frac{n}{m}\,.$$

Therefore $Q' = \frac{k}{\ell} \times \frac{n}{m}$, which is equal to Q.

The fact that for any fractions $\frac{m}{n}$ and $\frac{k}{\ell}$, with $\frac{m}{n} \neq 0$, there is a *unique* fraction Q so that

$$\frac{k}{\ell} = Q \times \frac{m}{n},$$

will be basic to the discussion of fraction division.

For example,

$$\frac{1}{12} = \left(\frac{1}{12} \times \frac{8}{117}\right) \times \frac{117}{8}$$

We now give a second interpretation of $\frac{k}{\ell} \times \frac{m}{n}$:

> $\frac{k}{\ell} \times \frac{m}{n}$ *is equal to* $\frac{k}{\ell}$ *copies of* $\frac{m}{n}$*, in the sense of everyday language.*

We now elaborate on this statement by considering, in succession, the case of $\frac{k}{\ell}$ being a whole number, a proper fraction and, finally, an **improper fraction** ($\frac{k}{\ell} > 1$).

If $\frac{k}{\ell}$ is a whole number, e.g., 5, then by the Product Formula,

$$5 \times \frac{m}{n} = \frac{5}{1} \times \frac{m}{n} = \frac{5m}{n} = \underbrace{\frac{m}{n} + \cdots + \frac{m}{n}}_{5},$$

which displays "5 copies of $\frac{m}{n}$".

If $\frac{k}{\ell}$ is a proper fraction, e.g., $\frac{3}{7}$, then by the definition of fraction multiplication, $\frac{3}{7} \times \frac{m}{n}$ is exactly "$\frac{3}{7}$ copies of $\frac{m}{n}$", i.e., not all of $\frac{m}{n}$, only 3 of the parts when $\frac{m}{n}$ is divided into 7 equal parts.

Finally, if $\frac{k}{\ell}$ is an improper fraction, e.g., $\frac{35}{4}$, then we write it as a mixed number, $8\frac{3}{4}$.

Suppose we have a bucket and its capacity is $\frac{m}{n}$ liters. Does $(8\frac{3}{4} \times \frac{m}{n})$ liters have the meaning of 8 and $\frac{3}{4}$ buckets ("$8\frac{3}{4}$ copies of $\frac{m}{n}$")?

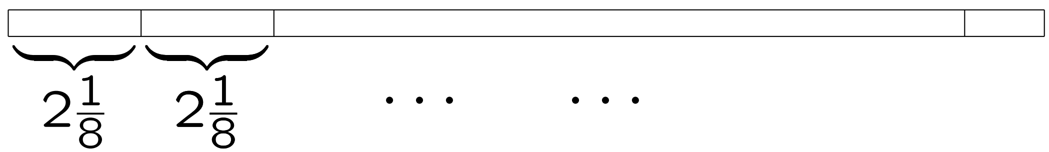
By the distributive law, $8\frac{3}{4} \times \frac{m}{n}$ liters is equal to

$$8\frac{3}{4} \times \frac{m}{n} = \left(8 + \frac{3}{4}\right) \times \frac{m}{n} = \left(8 \times \frac{m}{n}\right) + \left(\frac{3}{4} \times \frac{m}{n}\right)$$

Now $8 \times \frac{m}{n}$ liters is 8 buckets, and $\frac{3}{4} \times \frac{m}{n}$ liters is (by definition) $\frac{3}{4}$ of the bucket.

Thus $8\frac{3}{4} \times \frac{m}{n}$ liters is $8\frac{3}{4}$ buckets (if the capacity of the bucket is $\frac{m}{n}$ liters).

Example. *A rod* $15\frac{5}{7}$ *meters long is cut into short pieces which are each* $2\frac{1}{8}$ *meters long. How many short pieces are there?*



Students are taught that the way to do such problems is to divide $15\frac{5}{7}$ by $2\frac{1}{8}$, but without any explanation.

We will do it by applying the preceding ideas.

Let us say $\frac{a}{b}$ short pieces make up the rod. By what we just did, this says

$$\frac{a}{b} \times 2\frac{1}{8} \;=\; 15\frac{5}{7}\,,$$

i.e.,

$$\frac{a}{b} \times \frac{17}{8} \;=\; \frac{110}{7}$$

Multiplying both sides by $\frac{8}{17}$ (because we know $\frac{17}{8} \times \frac{8}{17} = 1$ from an earlier slide), we get

$$\left(\frac{a}{b} \times \frac{17}{8}\right) \times \frac{8}{17} \;=\; \frac{110}{7} \times \frac{8}{17}$$

so that $\displaystyle \frac{a}{b} \times \left(\frac{17}{8} \times \frac{8}{17}\right) \;=\; 7\frac{47}{119}$. Thus $\displaystyle \frac{a}{b} = 7\frac{47}{119}$.

Remark on $7\frac{47}{119}$:

We know the answer is: "there are $7\frac{47}{119}$ short pieces in the rod".

But what is the meaning of $\frac{47}{119}$ *in this context?*

In **TSM**, it is usually stated that it requires a *conceptual understanding of fraction* to see that the rod can yield 7 and $\frac{47}{119}$ *of a short piece.* We will show instead that what is required is a *mathematical* knowledge of the distributive law.

We are given that for the rod of length $15\frac{5}{7}$ meters,

$$15\frac{5}{7} \;=\; 7\frac{47}{119}\times 2\frac{1}{8}\;.$$

By the definition of mixed numbers,

$$\begin{aligned} 15\frac{5}{7} \;&=\; (7+\frac{47}{119})\times 2\frac{1}{8} \\ &=\; \left(7\times 2\frac{1}{8}\right) \;+\; \left(\frac{47}{119}\times 2\frac{1}{8}\right) \qquad\qquad (\text{dist. law}) \end{aligned}$$

This says, *explicitly*, that the whole rod consists of 7 short pieces, each being $2\frac{1}{8}$ meters long, plus $\frac{47}{119}$ *of a short piece* (because this is our definition of fraction multiplication).

Why not just define multiplication by the Product Formula:

$$\frac{k}{\ell} \times \frac{m}{n} \;=\; \frac{km}{\ell n}\,?$$

Because: (i) This definition of multiplication raises the question: why not define addition as

$$\frac{k}{\ell} + \frac{m}{n} \;=\; \frac{k+m}{\ell+n}$$

It may not be easy to explain to students why not.

(ii) Problems such as the one about the rod can only be done by rote if multiplication is defined by the product formula with no other meaning.

Multiplication of decimals.

In **TSM**, one multiplies two decimals, e.g., 0.00257×4.25, by multiplying the corresponding whole numbers, i.e., 257×425, then place the decimal point in the $(5+2)$th digit from the right. *No explanation.*

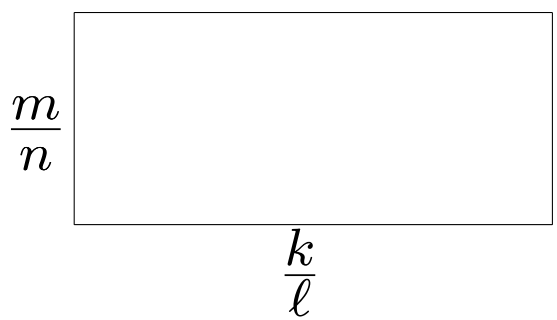
The reason is the Product Formula:

$$\begin{aligned} 0.00257 \times 4.25 &= \frac{257}{10^5} \times \frac{425}{10^2} \\ &= \frac{257 \times 425}{10^{5+2}} \quad \text{(product formula)} \\ &= \frac{109225}{10^{5+2}} = 0.0109225. \end{aligned}$$

Multiplication of fractions as area.

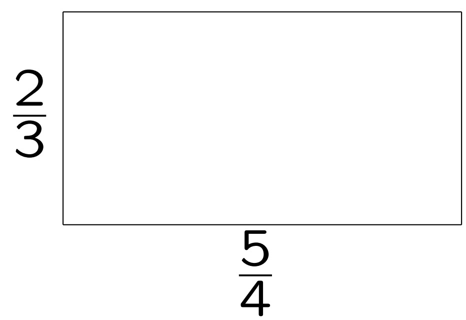
The following **Theorem** complements our understanding of what fraction multiplication means.

$$\boldsymbol{\frac{m}{n} \times \frac{k}{\ell}} \;=\; \textit{area of rectangle with sides } \frac{m}{n} \textit{ and } \frac{k}{\ell}$$



By the Product Formula, it suffices to prove that the area of a rectangle with sides $\frac{m}{n}$ and $\frac{k}{\ell}$ is $\frac{mk}{n\ell}$.

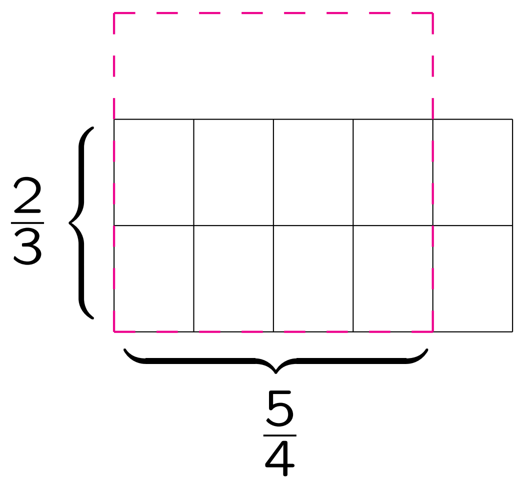
It suffices to give the proof of the theorem for the special case of a rectangle with sides $\frac{2}{3}$ and $\frac{5}{4}$, because the reasoning in the general case is no different. Thus we have:



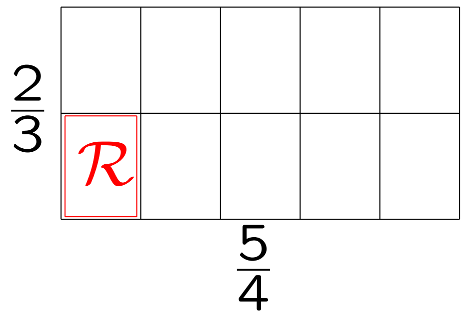
Will prove that this rectangle has area $\dfrac{2 \times 5}{3 \times 4}$.

Recall that $\frac{2}{3}$ is 2 copies of $\frac{1}{3}$, and $\frac{5}{4}$ is 5 copies of $\frac{1}{4}$. When we put the **unit square** (i.e., a square all of whose sides have length 1) in the background (the dashed magenta square), we see that

The vertical side of the rectangle is $\frac{2}{3}$ of the vertical side of the unit square, while the horizontal side is $\frac{5}{4}$ of the horizontal side of the unit square.

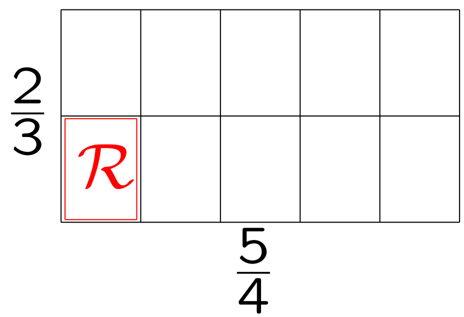


If we can find out the area of any of the small rect-angles, such as the red one $\mathcal{R}$ below, then the area of the big rectangle would just be the sum of (2×5) of the area of $\mathcal{R}$.



We will prove that the area of $\mathcal{R}$ is $\frac{1}{3 \times 4}$.

Once we know that area of $\mathcal{R}$ is $\frac{1}{3\times 4}$, then the area of the original rectangle

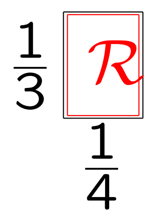


would be

$$\underbrace{\frac{1}{3\times 4}+\cdots+\frac{1}{3\times 4}}_{2\times 5} = \frac{2\times 5}{3\times 4},$$

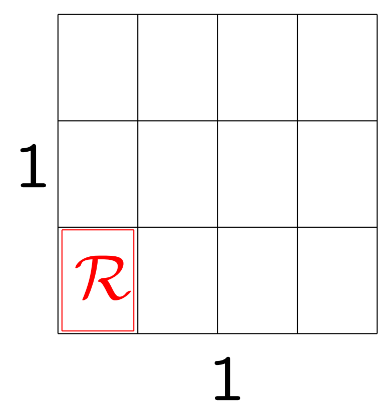
and the proof would be complete.

We now concentrate on proving that the area of $\mathcal{R}$ is $\frac{1}{3\times 4}$. Recall $\mathcal{R}$ has sides $\frac{1}{3}$ and $\frac{1}{4}$.



We can place $\mathcal{R}$ in the unit square as before:

Divide the unit square into 4 equal parts vertically, and 3 equal parts horizontally. Then the unit square is *paved* by 3×4 $(= 12)$ rectangles all of which have sides of lengths $\frac{1}{3}$ and $\frac{1}{4}$.



Observe that each of these 12 rectangles is congruent to $\mathcal{R}$, as shown.

On the number line, let the unit 1 be the area of the unit square. The areas of these 12 rectangles therefore provide a division of the unit 1 into 12 equal parts, so the area of any one of these 12 rectangles is

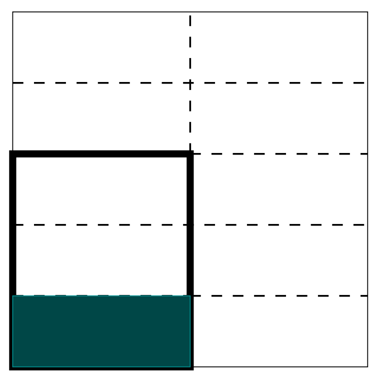
$$\frac{1}{12} = \frac{1}{3 \times 4}$$

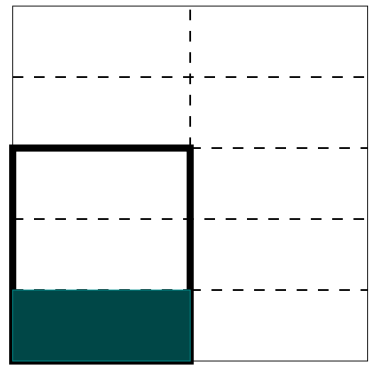
In particular, the area of $\mathcal{R}$ is $\frac{1}{3\times4}$. As we said before, this finishes the proof that a rectangle with sides $\frac{2}{3}$ and $\frac{5}{4}$ has area $\frac{2\times5}{3\times4}$.

Here is a problem for you:

Explain why the area of a rectangle with sides $\frac{3}{5}$ and $\frac{1}{2}$ is $\frac{3 \times 1}{5 \times 2}$.

Solution. Divide the unit square vertically into 5 equal parts and horizontally into 2 equal parts. By definition, we have to show that the area of the *thickened* rectangle is $\frac{3\times 1}{5\times 2}$.





Solution (cont.) The black (lower left) rectangle, being one of 10 ($= 5 \times 2$) congruent rectangles that pave the unit square, is 1 part when the unit is divided into 10 equal parts; so it has area $\frac{1}{5\times 2}$. Since 3 of these pave the thickened rectangle, the latter has area $\frac{3}{5\times 2} = \frac{3\times 1}{5\times 2}$.

We reduced the problem of computing the area of the $\frac{2}{3}$ by $\frac{5}{4}$ rectangle to that of computing the area of a $\frac{1}{3}$ by $\frac{1}{4}$ rectangle.

Where have you seen this kind of reasoning before?

According to the CCSSM:

5.NF 4b Find the area of a rectangle with fractional side lengths by tiling it with [rectangles] of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

PART III. DIVISION

To uncover the meaning of the division of fractions, again we look to whole numbers for guidance.

Because whole numbers are themselves fractions, their division cannot be conceptually different from the division of arbitrary fractions.

We tell students that $\frac{24}{6}$ (the preferred notation for "$24 \div 6$") is 4 because $4 \times 6 = 24$, that $\frac{48}{3} = 16$ because $16 \times 3 = 48$, that $\frac{54}{18} = 3$ because $3 \times 18 = 54$, etc.

In general, if m, n, q are whole numbers, $n \neq 0$ and m is a multiple of n, then we say

$$\frac{\boldsymbol{m}}{\boldsymbol{n}} = \boldsymbol{q} \quad \text{if} \quad m = qn$$

Three comments:

(i) Division among whole numbers is nothing more than a different way of writing a multiplication fact. This is true for the division of numbers in general.

(ii) A division $\frac{m}{n}$ among whole numbers m, n cannot be carried out unless m is a multiple of n. For example, $\frac{37}{16}$ is not a division among whole numbers.

(iii) Among whole numbers, there is a distinction between **division** and **division-with-remainder**.

Division (like addition, subtractions and multiplication) is a **binary operation**, in the sense that it associates a third number to two given numbers.

Division-with-remainder, on the other hand, is not a binary operation as it sends 37 and 16 to two numbers, 2 (quotient) and 5 (remainder).

Because for whole numbers m, n ($n \neq 0$), the definition of m divided by n is the whole number q so that $m = qn$, we simply imitate this definition for fractions:

Given fractions M, N, ($N \neq 0$), we define $\boldsymbol{M}$ **divided by** $\boldsymbol{N}$ to be the fraction Q so that $M = QN$.

But we must be careful!

For whole numbers m and n ($n \neq 0$), we define $\frac{m}{n}$ *only* when we know there is a (unique) whole number q so that $m = qn$.

For *any* fractions $\frac{k}{\ell}$ and $\frac{m}{n}$ ($\frac{m}{n} \neq 0$), we have now defined their divison $\frac{k/\ell}{m/n}$. *But do we know that there is a unique fraction Q so that $\dfrac{k}{\ell} = Q \times \dfrac{m}{n}$?*

Yes!

Recall an earlier fact: *Given any nonzero fraction* $\frac{m}{n}$ *and any fraction* $\frac{k}{\ell}$, *there is a unique fraction* Q *so that* $\dfrac{k}{\ell} = Q \times \dfrac{m}{n}$. In fact, $Q = \dfrac{k}{\ell} \times \dfrac{n}{m}$.

We now know that the definition of fraction division is completely sound.

Here is a reformulation:

The statement that the division of a fraction M by a nonzero fraction N is equal to Q, i.e., $\frac{M}{N} = Q$, is merely a different way of writing the *multiplicative* fact that $M = QN$ for a unique fraction Q.

The fact that there is always such a fraction Q is guaranteed.

To recapitulate: if Q is the division of a fraction $\frac{k}{\ell}$ by $\frac{m}{n}$, then

$$Q = \frac{k}{\ell} \times \frac{n}{m}$$

In other words, "to divide, you invert and multiply", i.e., invert $\frac{m}{n}$ to get $\frac{n}{m}$ and then use it to multiply $\frac{k}{\ell}$.

Invert-and-multiply is now seen to be a totally transparent skill. There is no reason to avoid it.

TSM's explanation of the division of fractions is limited to "division is the inverse operation of multiplication" plus some analogies.

Keep in mind that **TSM** does not even explain what multiplication is.

Ours is not to reason why, just invert and multiply.

According to the CCSSM:

6.NS 1 ... use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because 3/4 of 8/9 is 2/3. (In general, $(a/b) \div (c/d) = ad/bc$. ...)

Let us revisit an earlier problem: *A rod* $15\frac{5}{7}$ *meters long is cut into short pieces which are* $2\frac{1}{8}$ *meters long. How many short pieces are there?*

Let there be Q short pieces in the rod. Then by the interpretation of multiplication, $15\frac{5}{7} = Q \times 2\frac{1}{8}$. By the definition of division, this means

$$Q = \frac{15\frac{5}{7}}{2\frac{1}{8}} = \frac{\frac{110}{7}}{\frac{17}{8}} = \frac{110}{7} \times \frac{8}{7} = \frac{880}{119} = 7\frac{47}{119}.$$

Of course, we used the invert and multiply rule to compute the division.

Notice that we have **explained** why this problem can be done by **dividing** $15\frac{5}{7}$ by $2\frac{1}{8}$.

TSM does not.

Comments: The discussion of division is heavily dependent on a solid knowledge of multiplication. First, the fact that $\frac{M}{N}$ always makes sense depends on a fact proved about multiplication. In the solution of word problems, such as the last problem with the rod, the possibility of reasoning with the problem to get a solution again depends on a solid grounding in multiplication.

In mathematics, foundational knowledge is always critical.

Division of decimals.

It is immediately reduced to the *division of whole numbers*. For example, using invert-and-multiply,

$$\frac{2.114}{0.87} = \frac{\frac{2114}{1000}}{\frac{87}{100}} = \frac{211400}{87000}$$

Or, if you prefer:

$$\frac{2114}{870}$$

This conclusion is not satisfactory because we want an answer in the form of a decimal. *How to convert a division of whole numbers into a decimal?* We give a partial answer.

Consider $\frac{15}{32}$:

$$\frac{15}{32} = \left(\frac{15 \times 10^5}{32}\right) \times \frac{1}{10^5}$$

Using long division, we obtain $1500000 = (46875 \times 32) + 0$ so that

$$\frac{15}{32} = \frac{46875}{10^5} = 0.46875$$

Explore: is the use of 10^5 critical? Suppose we use 10^8:

$$\frac{15}{32} = \left(\frac{15 \times 10^8}{32}\right) \times \frac{1}{10^8}$$

Using long division, we obtain

$1500000000 = (468 \times 32) + 0$, so that

$$\frac{15}{32} = \frac{46875000}{10^8} = 0.46875000 = 0.46875$$

Same answer.

Suppose we use 10^3:

$$\frac{15}{32} = \left(\frac{15 \times 10^3}{32}\right) \times \frac{1}{10^3}$$

Using long division, we obtain $15000 = (468 \times 32) + 24$ so that

$$\begin{aligned}\frac{15}{32} &= \left(\frac{(468 \times 32) + 24}{32}\right) \times \frac{1}{10^3} \\ &= \frac{468}{10^3} + \left(\frac{24}{32} \times \frac{1}{10^3}\right) \\ &= 0.468 + \left(\frac{3}{4} \times \frac{1}{10^3}\right)\end{aligned}$$

Note that $\frac{3}{4} \times \frac{1}{10^3} = \frac{75}{100} \times \frac{1}{10^3} = 0.00075$. So we get 0.46875 again.

Another example: $\frac{218}{625}$.

$$\frac{218}{625} = \left(\frac{218 \times 10^5}{625}\right) \times \frac{1}{10^5}$$

By the long division algorithm,

$$21800000 = (34880 \times 625) + 0$$

Thus,

$$\frac{218}{625} = \frac{34880}{10^5} = 0.34880 = 0.3488$$

Here is a problem for you:

Convert $\frac{15}{64}$ to a decimal.

Solution.

$$\frac{15}{64} = \frac{15 \times 10^6}{64} \times \frac{1}{10^6}$$

Long division gives $15000000 = (234375 \times 64) + 0.$

Thus

$$\frac{15}{64} = \frac{234375}{10^6} = 0.234375$$

We have so far dealt with fractions that lead to *finite* decimals. In general, a fraction such as $\frac{8}{33}$ will lead to a repeating decimal basically by the same method.

In that case, however, the remainder in each long division will not be 0. But the remainders repeat, so the digits in the quotient also repeat.

PART IV. COMPLEX FRACTIONS

If M and N are fractions (N understood to be nonzero from now on), the division $\frac{M}{N}$ is also fraction. $\frac{M}{N}$ is called a **complex fraction**.

M and N will continue to be called the **numerator** and **denominator** of $\frac{M}{N}$.

Suppose A, B, C, D are whole numbers, then:

$$\begin{aligned}
\frac{C \cdot A}{C \cdot B} &= \frac{A}{B} \\
\frac{A}{B} \pm \frac{C}{D} &= \frac{AD \pm BC}{BD} \\
\frac{A}{B} \cdot \frac{C}{D} &= \frac{AC}{BD} \\
\frac{A/B}{C/D} &= \frac{AD}{BC}, \quad \text{etc.}
\end{aligned}$$

Question: Do these still hold for *complex fractions*, i.e., when A, . . . , D are *fractions*?

The answer is yes.

The explanation can be a routine chore (keep using invert-and-multiply) or short but abstract.

The main point is, however, that once you get past fraction division, the "fractions" you encounter are almost all complex fractions (not ordinary fractions), and you should make your students aware of the extension of these formulas to complex fractions.

TSM does not.

PART V. PERCENT

The main problem with the learning of *percent* is identical to the problem of learning fractions: there is *no definition* of percent in **TSM**.

What is a student supposed to make of

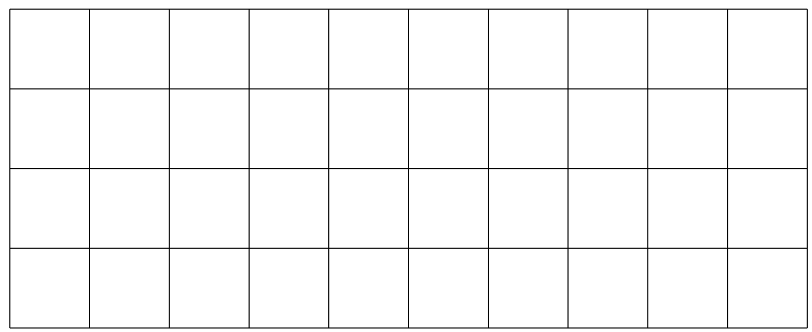
"out of a hundred"?

How to reason *precisely* with "out of a hundred"?

How to compute with "out of a hundred"?

Problem in a seventh grade class:[*]

`Shade 6 of the small squares in the rectangle shown below.`



`Using this diagram, explain how to determine the percent of the area that is shaded.`

[*]M.K. Stein, M.S. Smith,, M.A. Henningsen, E.A. Silver, *Implementing Standards-Based Mathematics Instruction*, Teachers College, Columbia University, 2000. P. 47.

The teacher's goal was for the students to figure out the percent representation of shaded portions of a series of rectangles.

He wanted his students to "use the visual diagrams to determine their numerical answers rather than relying on the traditional algorithms" that students had learned.

He wanted to help students develop "conceptual understandings of [this form] of representing fractional quantities. . ."

It turned out that, after 30 minutes, his students had no success.

Observe: there are 40 squares in the rectangle. The teacher wanted to know $\frac{6}{40}$ as a percent.

"Visual diagrams" may allow student to see $\frac{6}{40}$ as a percent, but what about $\frac{6}{39}$ as a percent? Would "out of a hundred" help? What kind of conceptual understanding are we talking about here?

Definition. *A* **percent** *is a complex fraction whose denominator is 100.*

A percent $\frac{N}{100}$, where N is a *fraction,* is often written as $N\%$. By regarding $\frac{N}{100}$ as an ordinary fraction, we recall that $\boldsymbol{N}$**% of a quantity** $\boldsymbol{\frac{m}{n}}$ is exactly $N\% \times \frac{m}{n}$. Now

$$N\% \times \frac{m}{n} \;=\; N \times \left(\frac{1}{100} \times \frac{m}{n}\right). \tag{1}$$

The expression $\frac{1}{100} \times \frac{m}{n}$ means the length of 1 part when $[0, \frac{m}{n}]$ is divided into 100 equal parts.

Therefore, equation (1) explains that,

> N% *of* $\frac{m}{n}$ *is* N *copies of a part when* $[0, \frac{m}{n}]$ *is divided into 100 equal parts.*

Equation (1) makes precise the naive concept of "percent" as "out of a hundred".

But equation (1) says more: even when N is a fraction (e.g., $\frac{3}{17}$), we can still make sense of "out of a hundred".

Next, by *strictly following the definition of* percent *and using the established facts about fraction multiplication*, we will do the following problems:

(i) What is 5% of 24?

(ii) 5% of what number is 16?

(iii) What percent of 24 is equal to 9?

(i) 5% of 24 is $5\% \times 24 = \frac{5}{100} \times 24 = \frac{6}{5}$.

(ii) Let us say that 5% of a certain number y is 16, then again strictly from the definition given above, this translates into $(5\%) \times y = 16$, i.e., $y \times \frac{5}{100} = 16$. By the definition of division, this says

$$y = \frac{16}{\frac{5}{100}} = 16 \times \frac{100}{5} = 320$$

(iii) Suppose $N\%$ of 24 is 9. This translates into $N\% \times 24 = 9$, or $\frac{N}{100} \times 24 = 9$. Multiplying both sides by $\frac{100}{24}$, we have

$$N = \frac{900}{24} = \frac{75}{2} = 37\frac{1}{2}$$

So the answer is $37\frac{1}{2}\%$.

We observe that without the precise definition of percent as a complex fraction, none of these solutions could have been obtained in a logical *manner.*

Let us return to the problem at the beginning.

There are 40 squares in the rectangle, and we must express 6 out of 40 as a percent, i.e.,

$$\text{if} \quad \frac{6}{40} = \frac{N}{100} \quad \text{for a fraction } N\text{, what is } N?$$

It is simple:

$$N = \frac{6 \times 100}{40} = 15$$

So the answer is 15%.

The teacher, however, had in mind something like this: *There are 40 squares, so 4 squares constitute 10%. Another 2 would therefore add 5%. As* $6 = 4 + 2$, *6 squares make up 15%.*

The teacher's solution may be cute, but it has *very* limited scope. For example, how would this method help you express 6 out of 39 squares as a percent?

Do you believe that the teacher's solution shows more conceptual understanding than the hard work we have done?

Let us express 6 out of 39 as a percent: if $\frac{6}{39} = N\%$ for some fraction N, then

$$\frac{6}{39} = \frac{N}{100}, \quad \text{so that} \quad N = \frac{6 \times 100}{39}.$$

(This is an equation between complex fractions.)

We have $N = 15\frac{15}{39}$.

Answer: 6 is $(15\frac{15}{39})\%$ of 39.

Here is a problem for you:

1 is what percent of 85?

Solution:

Let 1 be $N\%$ of 85, where N is a fraction. Then

$$1 \;=\; N\% \times 85 \;=\; \frac{N}{100} \times 85$$

Then $N = \frac{100}{85} = 1\frac{3}{17}$.

Answer: 1 is $(1\frac{3}{17})\%$ of 85.

PART VI. RATIO

The teaching of *ratio* suffers the same fate as the teaching of *percent*. Here is one of many similar explanations ("definitions") of *ratio* in **TSM**:

> A ratio is a comparison of any two quantities.
> A ratio may be used to convey an idea that cannot be expressed as a single number.

We cannot do mathematics on this basis. Let us give a precise definition of *ratio*.

Definition. *Given two fractions A and B. The* **ratio of A to B**, *sometimes denoted by* $\boldsymbol{A : B}$, *is the complex fraction* $\boldsymbol{\frac{A}{B}}$.

By convention, to say that **the ratio of boys to girls** in a classroom is 3 to 2 means that, if B (resp., G) is *the number of* boys (resp., girls) in the classroom, then $\frac{B}{G} = \frac{3}{2}$.

Similarly, in making a fruit punch, the statement that **the ratio of fruit juice to rum is 7 to 2** means that we are comparing the *volumes* of the two fluids (when the use of volume as the unit is understood in this situation).

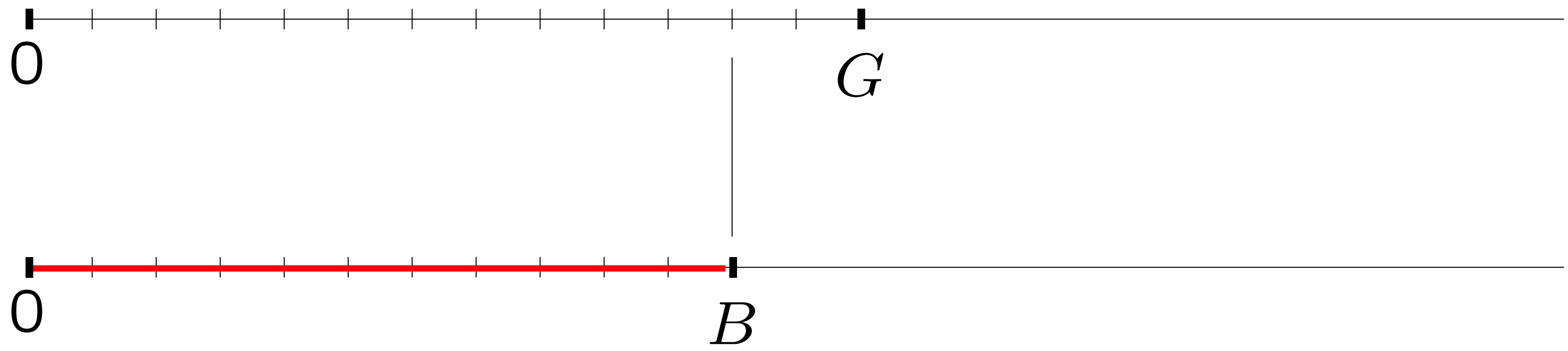
If the amount of fruit juice is A fluid ounces and the amount of rum is B fluid ounces, to say that **the ratio of fruit juice to rum is 7 to 2** means $\frac{A}{B} = \frac{7}{2}$.

Example. *In a school auditorium with 696 students, the ratio of boys to girls is 11 to 13. How many are boys and how many are girls?*

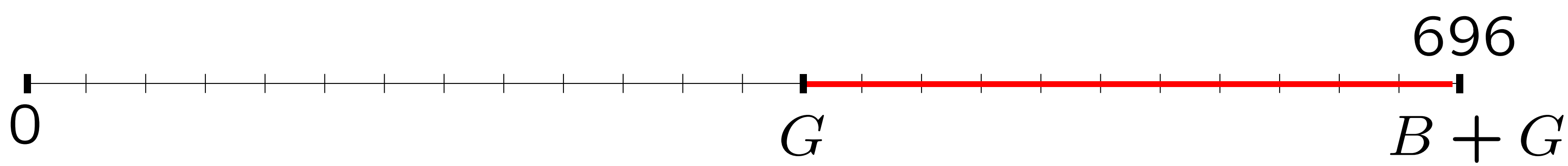
Let the number of boys be B and the number of girls be G, then we are given that $\frac{B}{G} = \frac{11}{13}$. Therefore,

$$B = \frac{11}{13} \times G$$

By the interpretation of multiplication, this says the number $B =$ the total number in 11 groups when the girls (G in toto) are divided into 13 equal groups.



Therefore:



Therefore the 696 students are now divided into $11+13$ equal groups, of which the girls comprise 13 of these groups and the boys 11. Since the size of one group is $\frac{696}{24} = 29$, we see that $G = 13 \times 29 = 377$ and $B = 11 \times 29 = 319$.

Although ratio is a division concept, the preceding reasoning leans heavily on our understanding of what *multiplication* means. One cannot overemphasize the fact that division is nothing but an alternate way of writing multiplication facts.

Additional comments. (i) This solution follows the definition of ratio, and the reasoning is *strictly by the book and completely straightforward. No need to guess what ratio means.*

(ii) The solution shows that the above definition of *ratio* is completely intuitive (even if it appears to be otherwise) and, because of its precision, lends itself easily to direct computations. The latter fact makes it learnable.

We now give a computational solution.

We are given that $\frac{B}{G} = \frac{11}{13}$. Thus by CMA, $13B = 11G$. Let k be this common number, i.e., $13B = 11G = k$, so $B = \frac{k}{13}$ and $G = \frac{k}{11}$.

Now we are also given $B + G = 696$, so $\frac{k}{13} + \frac{k}{11} = 696$. This gives $\frac{24k}{143} = 696$, and therefore $24k = 143 \times 696$, i.e., $k = 29 \times 143$. Since $B = \frac{k}{13}$, we get $B = 319$. The value of G can be obtained from either $B + G = 696$, or from $G = \frac{k}{11}$. In any case, $G = 377$.

Here is a problem for you:

In a town, two-thirds of the men are married to five-seventh of the women. What is the ratio of men to women?

Let M be the number of men and W be the number of women. Want $\frac{M}{W}$.

Given $\frac{2}{3} \times M = \frac{5}{7} \times W$. Using CMA, we get

$$\frac{M}{W} = \frac{\frac{5}{7}}{\frac{2}{3}} = \frac{15}{14}.$$

PART VII. CONSTANT RATE

In **TSM**, “rate” is regarded as a prominent concept. The standard “rates” are those arising from distance (**speed**), faucets (**rate of water flow**), lawns (**rate of lawn-moving**), and houses (**rate of house-painting**).

For simplicity, we will concentrate on speed.

Consider the following standard “rate problem”:

> Ellen walks 500 meters in 8 minutes. How far does she walk in $1\frac{1}{2}$ minutes?

According to **TSM**, there are at least two ways to solve this problem.

First, there is the common practice of “setting up a proportion”.

Students are told that if x is the number of meters Ellen walks after $1\frac{1}{2}$ minutes, then

“500 is to x (meters) as 8 is to $1\frac{1}{2}$ (minutes)”,

and therefore

$$\frac{500}{x} = \frac{8}{1\frac{1}{2}}$$

Solving for x by CMA, we get $x = 93\frac{3}{4}$ meters.

(This is an equality between complex fractions.)

A second method is to find the distance Ellen walks in *unit time*.

In this case, it is natural to choose the unit to be a half-minute (30 seconds). Since Ellen walks 500 meters in 16 units, she walks (500/16) meters per unit time, i.e., $31\frac{1}{4}$ meters each half minute.

Therefore in 3 units ($1\frac{1}{2}$ minutes), she walks

$$3 \times 31\frac{1}{4} \; = \; 93\frac{3}{4} \text{ meters.}$$

No reason is given for either method. Many students are known to be confused about why one can set up a proportion and why the problem can be done by choosing a unit time.

In fact, the problem as is cannot be solved.

For example, if Ellen walks in the way described below, then neither method would make sense.

Suppose Ellen walks in the following way in the first two minutes, and repeats it in subsequent two-minute intervals (she still walks 500 meters in 8 minutes):

time after she starts	*distance she walks*
first minute	100 meters
1 minute to $1\frac{1}{2}$ minutes	at rest
$1\frac{1}{2}$ minutes to 2 minutes	25 meters

Then clearly she walks 100 meters in the first $1\frac{1}{2}$ minutes, not $93\frac{3}{4}$ meters. Moreover, she walks 0 meters in the third unit time (from 1 minute to $1\frac{1}{2}$ minutes).

The implicit assumption, which **TSM** often takes for granted, is that Ellen walks **at constant speed**. Reformulation:

> Ellen walks at constant speed and she walks 500 meters in 8 minutes. How far does she walk in $1\frac{1}{2}$ minutes?

Unfortunately, one almost never finds a correct definition of *constant speed* in **TSM**.

To define constant speed, we begin with the more basic concept of the **average speed over a time interval from time s to time t**, $s < t$, as

$$\frac{\text{distance traveled from } s \text{ to } t}{t - s}$$

The term "average speed" by itself carries no *information because we have to know the average speed* from a specific point in time s to another point in time t.

In the example of Ellen walking during the first two minutes, let the unit of time t be *minutes* and let $t = 0$ at the start. Then her average speed from $t = 0$ to $t = 1$ is 100 m/min.

Her average speed from $t = 0$ to $t = 1.5$ is 100/1.5, which is $66\frac{2}{3}$ m/min, and her average speed from $t = 1.5$ to $t = 2$ is 50 m/min.

Her average speed is not the same over different time intervals.

Definition. A motion of **constant speed** $\boldsymbol{v}$ (v a fixed number) is one in which the average speed over *any* time interval is equal to v m/min.

The emphasis is on "*any*".

This concept will take 6th and 7th graders some time to get used to.

Ellen walks at constant speed and she walks 500 meters in 8 minutes. How far does she walk in $1\frac{1}{2}$ minutes?

Her average speed over the 8-minute interval is $\frac{500}{8}$ m/min.

Let Ellen walk x meters in $1\frac{1}{2}$ minutes. Her average speed over this interval is $\frac{x}{1\frac{1}{2}}$ m/min.

By the assumption of constant speed, $\dfrac{500}{8} = \dfrac{x}{1\frac{1}{2}}$.

Under the explicit assumption of constant speed, we can now understand why one can set up a proportion.

It is not "conceptual understanding about proportional reasoning." Rather, it is a *mathematical hypothesis* that must be explicitly given.

There are many gaps in **TSM**.

With constant speed, we also see why it is OK to pick *any* time interval to be the *unit time*.

Suppose we decide the *unit time* should be 30 seconds. Let Ellen walk k meters in unit time, then her average speed over unit time is $\frac{k}{1/2}$ m/min. Her average speed over 8 minutes is of course $\frac{500}{8}$ m/min.

Constant speed implies $\frac{k}{\frac{1}{2}} = \frac{500}{8}$ and $k = 31\frac{1}{4}$ m.

Suppose Ellen walks x meters in $1\frac{1}{2}$ minutes. Then her average speed is $\frac{x}{3/2}$ m/min.

By constant speed, $\dfrac{x}{\frac{3}{2}} = \dfrac{k}{\frac{1}{2}}$, so that

$$x \; = \; \left(\tfrac{3}{2}\right) \times \frac{k}{\left(\frac{1}{2}\right)} \; = \; 3 \times k \quad \text{meters,}$$

which is the same as before. (*Complex fractions!*)

In other settings, *constant speed* becomes *constant rate*. We illustrates with water flow.

The **average rate of water flow over a time interval from s to t**, $s < t$, is by definition

$$\frac{\text{total output of water from } s \text{ to } t}{t - s}$$

If water output is measured in gallons, the unit would be gal/min.

By definition, water flows at a **constant rate** if the average rate of water flow is a fixed number over any time interval.

We revisit the **Goals of these sessions:**

We give precise definitions for all concepts.

Especially *fractions*, *multiplication of fractions*, *division of fractions*, *percent*, etc.

We give a logical reason for every assertion we make.

We don't force students to believe anything that we cannot explain, *logically.*

Think of *invert and multiply*, why "two-thirds of five pounds" is $\frac{2}{3} \times 5$ pounds, why $0.0028 \times 0.543 = 0.0015204$, etc.)

Everything we do is consistent with the CCSSM.

We are trying to get out of the quagmire of **TSM**.